Integral ℓ-adic periods, convergent group rings and improvement of the results on boundedness and density.

Last time:

Thm (Existence of Bogomolov elements)

For any $\alpha \in \mathbb{Z}_\ell^\times$, sufficiently close to 1, there exists $\sigma_\alpha \in G_k$ s.t. σ_α acts on

$$Gr_W^{-i}\, \mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]$$

via $\alpha^i \cdot id$.

Rmk (1) Thm justifies the name of $W^\cdot$ to be a weight filtration

Recall What should call a weight filtration (over a f.g. char 0 field)

Given $\rho: G_k \to GL_n(\mathbb{Q}_\ell)$ s.t. ρ is mixed if there exists a filtration $W^\cdot$ s.t.

$$W^i / W^{i-1}$$

is pure of weight i.

$\rightsquigarrow$ Such a filtration is called a weight filtration of ρ.

(2) Thm implies that $\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]$ admits a set of σ_α-eigenvectors with dense span in the $\mathcal{I}$-adic topology.

Because

- $\mathbb{Q}[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]$ is complete wrt $\mathcal{I}$-adic topology ($=$ $W^\cdot$-adic topology)

- If there is a inverse system V_i of vector spaces, and $\sigma \curvearrowright V_i$ via a semisimply (e.g. $V_i := R/w^i$)
$\rightsquigarrow \underbrace{V := \varprojlim V_i}$ is dense in V.
eigenvectors of σ on

__Warning__ This density result does not hold for the integral version!

__Example__ (Failure of density in the integral case).

Suppose $X = \mathbb{G}_m$. $x \in X(k)$. Then as a G_k-module,

$$\pi_1^\ell(X_{\bar{k}}, \bar{x}) = \mathbb{Z}_\ell(1)$$

$\therefore$ $H^1(X_{\bar{k}}, \mathbb{Z}_\ell) = \mathbb{Z}_\ell(-1)$

and

$\mathrm{Hom}(\pi_1^\ell(X_{\bar{k}}, \bar{x}), \mathbb{Z}_\ell) = H^1(X_{\bar{k}}, \mathbb{Z}_\ell)$

Grétar: $\underset{\text{top generator of } \pi_1^\ell.}{\gamma} \longmapsto 1+T$

- $\mathbb{Z}_\ell[[\pi_1^\ell]] \xrightarrow{\sim} \mathbb{Z}_\ell[[T]]$, $\mathcal{I}^n = (T^n)$
- $\mathbb{Q}_\ell[[\pi_1^\ell]] \xrightarrow{\sim} \mathbb{Q}_\ell[[T]]$, $\mathcal{I}^n = (T^n)$.

For $\sigma \in G_k$.

$$\sigma(1+T) = (1+T)^{\chi(\sigma)}$$

where $\chi : G_k \to \mathbb{Z}_\ell^\times$ is the cyclotomic character.

- σ-eigenvectors

$$(\log(1+T))^n \in \mathbb{Q}_\ell[[T]] \qquad n \in \mathbb{Z}_{\geq 0}.$$

has eigenvalue $\chi(\sigma)^n$.

- The span of $(\log(1+T))^n$ is dense in the (T)-adic topology.

  bc: $$(\log(1+T))^n = \left(T - \frac{T^2}{2} + \frac{T^3}{3} - \dots\right)^n$$

  has leading term T^n.

- If $\chi(\sigma) \neq 1$, the only σ-eigenvector in $\mathbb{Z}[[T]]$ is $\underline{1}$.

  $\rightsquigarrow$ σ-e.v. in $\mathbb{Z}[[T]]$ is far away from being (T)-adically dense !!!

1. Integral ℓ-adic periods

1.1 Heuristics

k $\quad X/k$ smooth $\quad \overline{X}/k$ SNC compadification, smooth

x is either a rational point or a rational tangental basepoint.

- Rational case: whenever $\sigma \in G_k$ is a Bogomolov element, the σ-action on $\mathbb{Q}_\ell[[\pi_1^\ell]]$ splits along the weight filtration.

  $\rightsquigarrow$ have a complete description of the action

  $$\mathbb{Q}_\ell[\sigma] \curvearrowright \mathbb{Q}_\ell[[\pi_1^\ell]].$$

- Integral case: σ-equivariant Spliting property fails on $\mathbb{Z}_\ell[[\pi_1^\ell]]$.

Aim: Examine this failure & measure how far it can be from being σ-equivarantly splitting.

Example (Failure of integrally splitting).

Let $r \in \mathbb{N}_{>0}$.

Consider

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}_\ell \arrow[r, "\binom{1}{0}"] \arrow[d] & \mathbb{Z}_\ell^2 \arrow[r, "(0\ 1)"] \arrow[d, "\begin{pmatrix} 1 & 1 \\ 0 & 1+\ell^r \end{pmatrix} = \sigma"] & \mathbb{Z}_\ell \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathbb{Z}_\ell \arrow[r] & \mathbb{Z}_\ell^2 \arrow[r] & \mathbb{Z}_\ell \arrow[r] & 0 \end{tikzcd}$$

$\rightsquigarrow$ This is a ses. of $\mathbb{Z}_\ell[\sigma]$-modules.

On tensoring $\mathbb{Q}_\ell$, it splits: via the map

$$\mathbb{Q}_\ell \to \mathbb{Q}_\ell^2 \quad \text{given by} \quad \begin{pmatrix} \frac{1}{\ell^r} \\ 1 \end{pmatrix}$$

But does not split integrally.

Def (Integral ℓ-adic period). Let

$$0 \to W \to V \xrightarrow{\pi} V/W \to 0$$

be a ses of free $\mathbb{Z}_\ell$-modules, and

$$\sigma : V \to V$$

is a $\mathbb{Z}_\ell$-endomorphism preserving W.

Define the integral ℓ-adic period $b_{W,V}$ of the triple $(W \subset V, F)$ to be

$$b_{W,V} := \inf \left\{ v_\ell(a) \in \mathbb{Z} \;\middle|\; \begin{array}{l} a \in \mathbb{Q}_\ell \text{ s.t. there exists a } \sigma\text{-equiv} \\ s : V/W \to V \text{ with } \pi \circ s = a \cdot \mathrm{id} \end{array} \right\}$$

If no such an a exists, set

$$b_{W,V} = \infty.$$

<u>Rmk</u>. i.e. we are just taking the minimum ℓ-adic valuation of a s.t.

$$\begin{tikzcd} & & & V/W \arrow[dl, "s"'] \arrow[d, "ⓐ"] & \\ 0 \arrow[r] & W \arrow[r] & V \arrow[r, "\pi"] & V/W \arrow[r] & 0 \end{tikzcd}$$

- The small $b_{W,V}$ is, the closer this sequence from being "integrally σ-equiv split".

<u>Example</u>. • $b_{W,V} = 0 \iff$ the sequence split integrally.

- Example at the beginning of this section $\rightsquigarrow b_{W,V} = r$

<u>Rmk</u> ("ℓ-adic Hodge theory").

If $b_{W,V}$ is finite, a choice of a "quasi-splitting" s induces

$$\tilde{s} : (W \oplus V/W) \otimes \mathbb{Q}_\ell \xrightarrow{\sim} V \otimes \mathbb{Q}_\ell.$$

The entries of the matrix asso. to $\tilde{s}$ have valuation $\geq -b_{W,V}$.

$\rightarrow$ view $\tilde{s}$ as a "comparison isom" analogus to p-adic Hodge theoretic comparison isom's.

<u>1.2</u> how to study it?

<u>A</u>: via $\mathrm{Ext}^1_{\mathbb{Z}_\ell[G]}$!

<u>Proposition</u> ① $V =$ f.g. free $\mathbb{Z}_\ell$-module.

$W \subset V$ free submodule.

$\sigma : V \to V$ preserving W.

Suppose the ses of $\mathbb{Z}_\ell[\sigma]$-modules.

$$0 \to W \to V \to V/W \to 0$$

corresponds to a class

$$\alpha \in \mathrm{Ext}^1_{\mathbb{Z}_\ell[\sigma]}(V/W, W).$$

Then

- If α is not $\mathbb{Z}_\ell$-torsion, then $\underline{b_{W,V}}$ is infinite.
- If α is $\mathbb{Z}_\ell$-torsion,

$$b_{W,V} = \inf \{ v_\ell(b) \mid b \cdot \alpha = 0 \text{ with } b \in \mathbb{Z}_\ell \}$$

1.3 Int ℓ-adic periods of π_1

Recall (weight filtration construction)

(1). $\mathbb{Q}_\ell$-weight filtration: For $i \le 0$ set

$$W^{-i} \mathbb{Q}_\ell[[\pi_1^\ell]] = \mathbb{Q}_\ell[[\pi_1^\ell]]$$

For $i > 0$

$$W^{-i}\mathbb{Q}_\ell[[\pi_1^\ell]] = I \cdot W^{-i+1}_{\mathbb{Q}_\ell} + J \cdot W^{-i+2}_{\mathbb{Q}_\ell}.$$

where $I^n_{\mathbb{Q}_\ell} := \ker\left(\mathbb{Q}_\ell[[\pi_1^\ell]] \to \mathbb{Z}_\ell[[\pi_1^\ell]]/I^n_{\mathbb{Z}_\ell} \otimes \mathbb{Q}_\ell\right)$

$J := \ker\left(\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}})]] \to \mathbb{Q}_\ell[[\pi_1^\ell(\bar{X}_{\bar{k}})]]\right)$

$\rightsquigarrow \mathbb{Q}_\ell[[\pi_1^\ell]]$ complete wrt I-adic topology.

Def (Star condition) In $\mathbb{Z}_\ell[[\pi_1^\ell]]$

$\star$ $\quad I^n_{\mathbb{Z}_\ell} / I^{n+1}_{\mathbb{Z}_\ell}$ is $\mathbb{Z}_\ell$-torsion free for all n.

Example ☆ holds when X is a curve

Rmk With ☆ condition

- $\mathbb{Z}_\ell[[\pi_1^\ell]] \to \mathbb{Q}_\ell[[\pi_1^\ell]]$ is injective.

  $\rightsquigarrow W^{-n}\mathbb{Z}_\ell[[\pi_1^\ell]] = \mathbb{Z}_\ell[[\pi_1^\ell]] \cap W^{-n}\mathbb{Q}_\ell[[\pi_1^\ell]]$
- $Gr^{-n}_{W,\mathbb{Z}_\ell}$ are torsion free for all n.

Thm (Bound of the int ℓ-adic period for Bogomolov σ_α)

k. X/k sm. $\overline{X}/k$ sm comp SNCB.

x either a rational point of X or a rational tangental base point of X.

Suppose.

- $\sigma \in G_k$ is a Bogomolov element of degree $\alpha^{-1} \in \mathbb{Z}_\ell^\times$ wrt the weight filtration on $\mathbb{Q}_\ell[[\pi_1^\ell]]$.
- X satisfies ☆ condition.

Then the integral ℓ-adic period $b_{i,n}$ asso. to the sequence of σ-modules

"integral"

$$0 \to W^{-i-1}_{\mathbb{Z}_\ell}/W^{-n}_{\mathbb{Z}_\ell} \to W^{-i}_{\mathbb{Z}_\ell}/W^{-n}_{\mathbb{Z}_\ell} \to W^{-i}_{\mathbb{Z}_\ell}/W^{-i-i}_{\mathbb{Z}_\ell} \to 0$$

satisfies

$$\boxed{b_{i,n} \leq C_{\alpha,\ell,n-i-1}}$$

Here $C_{\alpha,\ell,n}$ is a constant asso to α, ℓ, n.

Pf. Consequence of prop (1) + some calculation on valuation.

↳ next session.

2. Boundedness & Density

Aim. • Extendability : Introduce certain Gal-stable subobjects of of

$$\mathbb{Q}_\ell[[\pi_1^\ell]]$$

on which G_k acts.

$\to$ convergent group ring

$$\mathbb{Q}_\ell[[\pi_1^\ell]]^{\leq \ell^{-r}}$$

- Semisimplicity: In good cases, Bogomolov elements in G_k act diagonalizably on $\mathbb{Q}_\ell[[\pi_1^\ell]]^{\leq \ell^{-r}}$
- Density Thm (Christoph) In $\mathbb{Q}_\ell[[\pi_1^\ell]]$, the σ_α-e.v. has dense span in I-adic (thus W-adic) topology where σ_α is Bogomolov.

  $\to$ This will be refined in the final thm.

  (But! after changing of topology)

2.1 Valuation, convergent gp ring with radius.

Def. Define $v_n : \mathbb{Q}_\ell[[\pi_1^\ell]]/I^n \to \mathbb{Z} \cup \{\infty\}$

By: if $g \in \mathbb{Q}_\ell[[\pi_1^\ell]]/I^n$,

$$v_n(g) = -\inf\{ v_\ell(s) \mid s \in \mathbb{Q}_\ell \text{ s.t. } s \cdot g \in \mathbb{Z}_\ell[[\pi_1^\ell]]/I^n \}.$$

Rmk (1) The smaller $v_\ell(s)$ is, the closer g is towards being integral.

$\Leftrightarrow$ The bigger $v_n(g)$ is,

(2) v_n is a non-arch val.

(3). $\pi_n : \underset{\ni g}{\mathbb{Q}_\ell[[\pi_1^\ell]]} \twoheadrightarrow \mathbb{Q}_\ell[[\pi_1^\ell]]/I^n$

Then

(2). $\pi_n : \underbrace{\mathbb{Q}_\ell[[\pi_1]]}_{\ni g} \twoheadrightarrow \mathbb{Q}_\ell[[\pi_1]]/I^n$

Then

$$\underline{v_n(\pi_n(g))} \geq v_{n+1}(\pi_{n+1}(g))$$

<u>Def</u>. $\underline{\mathbb{Q}_\ell[[\pi_1^\ell]]^{\leq \ell^{-r}}} := \{ g \in \underline{\mathbb{Q}_\ell[[\pi_1^\ell]]} \mid v_n(\pi_n(g)) + nr \to \infty \text{ as } n \to \infty \}$.

<u>convergent gp ring (of ~~radius~~ ℓ^{-r}</u>)

misleading.

Weight filtration ✓

<u>Topology</u> : Gauss norm (asso. to $r \in \mathbb{R}_{>0}$)

$$|g|_r := \sup_n \{ \ell^{-v_n(\pi_n(g)) - nr} \}$$

<u>Rmk</u> . (1). $r_1 < r_2$

$$\mathbb{Q}_\ell[[\pi_1^\ell]]^{\leq \ell^{-r_1}} \subset \mathbb{Q}_\ell[[\pi_1^\ell]]^{\leq \ell^{-r_2}}$$

(2). $\mathbb{Q}_\ell[[\pi_1^\ell]]^{\leq \ell^{-r}}$ are rings.

(3) $G_k \curvearrowright \mathbb{Q}_\ell[[\pi_1^\ell]]^{\leq \ell^{-r}}$

<u>Example</u> 3 π_1^ℓ is a free pro-ℓ gp. gen by

$$\gamma_1, -, \gamma_m.$$

- $\mathbb{Q}_\ell[[\pi_1^\ell]] \xrightarrow{\simeq} \mathbb{Q}_\ell\langle\langle T_1, -, T_m \rangle\rangle \quad \gamma_i \mapsto 1 + T_i$.
- $I \subset \mathbb{Q}_\ell[[\pi_1^\ell]]$ corresp to $(T_1, -, T_m)$
- The valuation / Gauss norm on $\mathbb{Q}_\ell[[\pi_1^\ell]]/I^n$ induces a val / Gauss norm on

$$\mathbb{Q}_\ell \langle\langle T_1, -, T_n \rangle\rangle / (T_1, -, T_n)^n$$

$$\rightarrow \quad \mathbb{Q}_\ell[[\pi_1^\ell]]^{\leq \ell^{-r}} = \{ \sum a_I T^I \mid \lim_{|I|\to\infty} v_\ell(a_I) + r|I| = \infty \}$$

$$\overset{?}{=} \{ \sum a_I T^I \mid |a_I|_\ell < \ell^{r|I|} \text{ for } I \text{ large enough} \}$$

<u>Fact</u> $\mathbb{Q}_\ell[[\pi_1^\ell]]^{\ell^{-r}} = \widehat{\mathbb{Z}_\ell[[\pi_1^\ell]] \otimes \mathbb{Q}_\ell}^{|\cdot|_r}$

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<u>Proposition</u> (Extension of integral representations)

Let

$$\rho : \pi_1^\ell := \pi_1^\ell(X_{\bar{k}}, \bar{x}) \to GL_n(\mathbb{Z}_\ell)$$

be a continuous rep. Suppose

- π_1^ℓ is a f.g. free pro-ℓ group by m generators
- ρ is trivial mod ℓ^m

<u>Then</u> $\forall\, r \in \mathbb{R}$ w/ $0 < r < m$, there exists a <u>unique</u> continuous ring hom

$$\tilde{\rho} : \mathbb{Q}_\ell[[\pi_1^\ell]]^{\leq \ell^{-r}} \to M_n(\mathbb{Q}_\ell)$$

making the diagram commute

$$\begin{array}{ccc} \mathbb{Z}_\ell[[\pi_1^\ell]] & \xrightarrow{\rho} & M_n(\mathbb{Z}_\ell) \\ \downarrow & & \downarrow \\ \mathbb{Q}_\ell[[\pi_1^\ell]]^{\leq \ell^{-r}} & \xrightarrow{\tilde{\rho}} & M_n(\mathbb{Q}_\ell) \end{array}$$

Pf. <u>Uniqueness</u> Last fact in <u>Example 3</u>

$$\widehat{\mathbb{Z}_\ell[[\pi_1^\ell]] \otimes \mathbb{Q}_\ell}^{|\cdot|_r} = \mathbb{Q}_\ell[[\pi_1^\ell]]^{\leq \ell^{-r}}$$

$\rightsquigarrow$ uniqueness ✓

<u>Existance</u> Define norm of a matrix $M \in M_n(\mathbb{Q}_\ell)$

Sup norm of matrix $M \in M_n(\mathbb{Q}_\ell)$

$$|M| := \ell^{\inf\{r \in \mathbb{Z} \mid \ell^r \cdot M \in M_n(\mathbb{Z}_\ell)\}}$$

$\rightsquigarrow$ $|\cdot|$ measures the distance of M towards being integral:
(The bigger $|M|$ is, the further M is from being integral)

$\hookrightarrow$ consistant with the intuition of the ℓ-adic norm

For $a \in \mathbb{Q}_\ell$, $|aM| = |a|_\ell \cdot |M|$

Continue with Example 3: topological generators

$$\lambda_1, \dots, \lambda_m$$

of $\mathbb{Z}_1^\ell$, and set

$$T_i = \lambda_i - 1 \in \mathbb{Q}_\ell[[\mathbb{Z}_1^\ell]]^{\le \ell^{-r}}$$

Define $\tilde{\rho} : \mathbb{Q}_\ell[[\mathbb{Z}_1^\ell]] \to M_n(\mathbb{Q}_\ell)$ by

$$\tilde{\rho}\left(\sum a_I T^I\right) := \sum_I a_I \, \rho(T^I)$$

with $a_I \in \mathbb{Q}_\ell$

Convergence: Since ρ is trivial mod ℓ^m $\longrightarrow$ $|\rho(T_i)| \le \ell^{-m}$

$\longrightarrow$ $|\rho(T^I)| = |\rho(T_1^{i_1} \cdots T_m^{i_m})| \le \ell^{-m(i_1 + \dots + i_m)} = \ell^{-m|I|}$

$\rightsquigarrow$ $|\tilde{\rho}(a_I T^I)| = |a_I \, \rho(T^I)| \le \underbrace{|a_I|_\ell}_{\boxed{< \ell^{r|I|} \text{ for } |I| \text{ large enough (Example 3)}}} |\rho(T^I)| \le \ell^{(r-m)|I|}$

Since $r < m$ $\rightsquigarrow$ $|\tilde{\rho}(a_I T^I)| \to 0$ when $|I| \to \infty$.

Continuity WTS There exists some $C > 0$ s.t.

$$\left|\tilde{\rho}\left(\sum a_I T^I\right)\right| \le C \left|\sum a_I T^I\right|_r$$

To this end,

$$\left|\tilde{\rho}\left(\sum a_I T^I\right)\right| \overset{\text{non-arch?}}{\le} \sup_I |a_I|_\ell \cdot |\tilde{\rho}(T^I)|$$

$$\le \sup_I \ell^{-v_\ell(a_I) - m|I|}$$

$$\le \sup_I \ell^{-v_\ell(a_I) - r|I|} \quad \text{(bc } r < m\text{)}$$

$$= \left| \sum a_I T^I \right|_r$$

i.e. we can take $C = 1$. $\boxed{\lightning}$

2.3 Boundedness & Density

<u>Theorem</u> (Litt 18, Thm 3.6) Let $\alpha \in \mathbb{Z}_\ell^\times$ be close enough to 1 & not a root of unity. Let σ_α be a Bogomolov element of degree α (as introduced by Christoph).

<u>Suppose</u>

- X satisfies condition ($\star$)

<u>Then</u> $\exists\, r_\alpha > 0$ s.t. for $r > r_\alpha$ (i.e. big enough disk)

(1) (Boundedness) every σ_α-eigenvector in $\mathbb{Q}_\ell[[\pi_1^\ell]]$ in fact lies in $\mathbb{Q}_\ell[[\pi_1^\ell]]^{\le \ell^{-r}}$

(2) (Density) The σ_α-eigenvectors in $W^{-i}\mathbb{Q}_\ell[[\pi_1^\ell]]^{\le \ell^{-r}}$ are dense in the Gauss norm topology

Proof of Theorem 6.13. We claim that we may take $r_\alpha = 2C(\alpha, \ell, 1)$, defined as in Lemma 6.4 (or Theorem 5.15).

(1) **Boundedness of σ_α-eigenvectors.**

Claim 1. For $r > r_\alpha$, every σ_α-eigenvector in $\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]$ in fact lies in $\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]^{\le \ell^{-r}}$. (i.e., the set of σ_α-eigenvector in $\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]$ is bounded.)

Claim 2. There exist unique σ_α-equivariant splittings s_i of the quotient map

$$W^{-i}\mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]^{\le \ell^{-r}} \to \mathrm{Gr}_W^{-i}\, \mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]],$$

for any $r > r_\alpha$.[33]

From Claim 2 to Claim 1. Now let $y \in \mathbb{Q}_\ell[[\pi_1^\ell(X_{\bar{k}}, \bar{x})]]$ be a non-zero σ_α-eigenvector; then there exists some maximal i such that $y \in \mathcal{I}^i$ and thus $y \in W^{-i}$ by Proposition 1.17. Then we have

$$\sigma_\alpha y = \alpha^i y \text{ in } \mathrm{Gr}_W^{-i}$$

because $y \notin W^{-i-1}$. Now we claim

$$y = s_i(\bar{y}) \text{ in } W^{-i}$$

where $\bar{y} \in W^{-i}/W^{-i-1}$ is the residue class of $y \in W^{-i}$. Indeed, $y - s_i(\bar{y})$ is a σ_α-eigenvector with eigenvalue α^i, contained in W^{-i-1} (because it is zero in

[33] I think there is a typo in the original paper here. See **Step 3** and **From Claim 2 to Claim 1**.

Gr_W^{-i}), hence zero because σ_α is a Bogomolov element (cf. Theorem 4.1). Thus the claim holds and

$$\sigma_\alpha y = \alpha^i y \text{ in } W^{-i}\mathbb{Q}_\ell[[\pi_1^\ell(X_{\overline{k}}, \overline{x})]]^{\leq \ell^{-r}}$$

where we have used that s_i is σ_α-equivariant (and $\mathbb{Q}_\ell$-linear).

Sketch of the Claim 2.

Step 1 (Graded splitings s_i^m).

By Corollary 5.8(-typed computation), there *exists* a σ_α-equivariant map

$$s_i^m : \mathrm{Gr}_W^{-i}\,\mathbb{Z}_\ell[[\pi_1^\ell(X_{\overline{k}}, \overline{x})]] \to W^{-i}\mathbb{Z}_\ell[[\pi_1^\ell(X_{\overline{k}}, \overline{x})]]/W^{-m}\mathbb{Z}_\ell[[\pi_1^\ell(X_{\overline{k}}, \overline{x})]]$$

such that the diagram

$$\begin{tikzcd}
 & & & \mathrm{Gr}_W^{-i}\,\mathbb{Z}_\ell[[\pi_1^\ell(X_{\overline{k}}, \overline{x})]] \arrow[dl, "s_i^m"'] \arrow[d, "\cdot \ell^{v(i,m,\alpha)}"] & \\
0 \arrow[r] & W^{-i-1}/W^{-m} \arrow[r] & W^{-i}/W^{-m} \arrow[r] & \mathrm{Gr}_W^{-i}\,\mathbb{Z}_\ell[[\pi_1^\ell(X_{\overline{k}}, \overline{x})]] \arrow[r] & 0
\end{tikzcd}$$

commutes, where W^{-i} above denotes $W^{-i}\mathbb{Z}_\ell[[\pi_1^\ell(X_{\overline{k}}, \overline{x})]]$, and

$$v(i,m,\alpha) = \sum_{s=1}^{m-i-1} v_\ell(\alpha^s - 1).$$

Moreover s_i^m is claimed to be *unique*.

Step 2 (Extension to rational coefficients by $\mathcal{I}$-completeness). By the above uniqueness, all these s_i^m form a inverse system (with proper connecting homomorphisms). That is, the diagram

$$\begin{tikzcd}
 & \mathrm{Gr}_W^{-i}\,\mathbb{Z}_\ell[[\pi_1^\ell(X_{\overline{k}}, \overline{x})]] \arrow[dl, "\ell^{-v(i,m+1,\alpha)}\cdot s_i^{m+1}"'] \arrow[d, "\ell^{-v(i,m,\alpha)}\cdot s_i^m"] \\
W^{-i}\mathbb{Q}_\ell[[\pi_1^\ell(X_{\overline{k}}, \overline{x})]]/W^{-m-1}\mathbb{Q}_\ell[[\pi_1^\ell(X_{\overline{k}}, \overline{x})]] \arrow[r] & W^{-i}\mathbb{Q}_\ell[[\pi_1^\ell(X_{\overline{k}}, \overline{x})]]/W^{-m}\mathbb{Q}_\ell[[\pi_1^\ell(X_{\overline{k}}, \overline{x})]]
\end{tikzcd}$$

commutes. (Note that we have extended the scalars in the image so as to have ℓ^{-1}). Since $\mathbb{Q}_\ell[[\pi_1^\ell(X_{\overline{k}}, \overline{x})]]$ is complete with respect to the $\mathcal{I}$-adic, hence $W^\bullet$ filtration, extending scalars of the source, the maps

$$\{\ell^{-v(i,m,\alpha)} \cdot s_i^m\}_{m>i}$$

define a σ_α-equivariant map

$$\widetilde{s}_i : \mathrm{Gr}_W^{-i}\,\mathbb{Q}_\ell[[\pi_1^\ell(X_{\overline{k}}, \overline{x})]] \to W^{-i}\mathbb{Q}_\ell[[\pi_1^\ell(X_{\overline{k}}, \overline{x})]],$$

splitting the natural quotient map

$$W^{-i}\mathbb{Q}_\ell[[\pi_1^\ell(X_{\overline{k}}, \overline{x})]] \twoheadrightarrow \mathrm{Gr}_W^{-i}\,\mathbb{Q}_\ell[[\pi_1^\ell(X_{\overline{k}}, \overline{x})]]$$

Step 3. This map factors through $\mathbb{Q}_\ell[[\pi_1^\ell(X_{\overline{k}}, \overline{x})]]^{\leq \ell^{-r}}$ for $r > 2C(\alpha, \ell, 1)$, giving the desired maps

$$s_i : \mathrm{Gr}_W^{-i}\,\mathbb{Q}_\ell[[\pi_1^\ell(X_{\overline{k}}, \overline{x})]] \to W^{-i}\mathbb{Q}_\ell[[\pi_1^\ell(X_{\overline{k}}, \overline{x})]]^{\leq \ell^{-r}}.$$

The proof of this step essentially uses the upper bound of the integral ℓ-adic period Theorem 5.15. Proof omitted. □

(2) **Density of σ_α-eigenvectors.**

We want to prove: the σ_α-eigenvectors in $W^{-i}\mathbb{Q}_\ell[[\pi_1^\ell(X_{\overline{k}}, \overline{x})]]^{\leq \ell^{-r}}$ are dense in the topology defined by the Gauss norm.

"Gauss-Approximation": Given $z \in W^{-i}\mathbb{Q}_\ell[[\pi_1^\ell(X_{\overline{k}}, \overline{x})]]^{\leq \ell^{-r}}$, let

$$z_0 = z, w_0 = s_i(z_0 \bmod W^{-i-1})$$

and in general,

$$z_j = z_{j-1} - w_{j-1}, w_j = s_{i+j}(z_j \bmod W^{-i-j-1}).$$

Then the claim is that [34]

$$\sum z_j \to z.$$

The proof of this needs some estimates of valuations (including Lemma 6.4) and we omit it.

□

Remark 6.14. If $H^1(X_{\overline{k}}, \mathbb{Q}_\ell)$ is pure of weight i ($i = 1, 2$), we may take $r_\alpha = C(\alpha^i, \ell, 1)$ (because in this case the weight filtration equals the $\mathcal{I}$-adic filtration, up to renumbering).